Intelligent OFDM telecommunication system. Part 2. Examples of many-parameter transforms

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Abstract. In this paper, we propose unified mathematical forms of many-parametric Fourier and wavelet transforms (MPFT and MPWT) for novel Intelligent OFDM-telecommunication systems (OFDM-TCS). Each many-parametric transform (MPT) depends on many free angle parameters. When parameters are changed in some way, the type and form of transform are changed as well. For example, MPT may be the Fourier transform for one set of parameters, wavelet transform for other parameters and other transforms for other values of parameters. The new Intelligent-OFDM-TCS use inverse MPT for modulation at the transmitter and direct MPT for demodulation at the receiver.

1. Introduction

One of the best-unknown MPT was developed by the 19^{th} century mathematician Jacobi [1]. The Jacobi algorithm composes method for computing the eigenvalue decomposition of symmetric matrix or for many-parameter representation of an orthogonal matrix $\mathbf{U}$ [2]-[3]. We recall that Jacobi's sequential method (Jacobi cyclic row algorithm (JCRA)) reduces an orthogonal matrix $\mathbf{U}$ to identical matrix by applying orthogonal rotations to right of $\mathbf{U}$, $\mathbf{Q}=\mathbf{U}\cdot\mathbf{J}\left(\varphi_{pq}\right)$, where orthonormal Jacobi rotation with reflection

$$\mathbf{J}(\varphi_{p,q}) = \begin{array}{cc} & \begin{array}{ccccccc} & & p & & q & & \end{array} \\ \begin{array}{c} \\ \\ p \\ \\ q \\ \\ \\ \end{array} & \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & s & \cdots & -c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix} \end{array}, \qquad (1)$$

is used to reduce the element U_{pq} or A_{pq} to zero. Jacobi rotation $\mathbf{J}\left(\varphi_{pq}\right)$ operates on *p*-th and *q*-th element of the *p*-th row of $\mathbf{U}$, U_{pp} and U_{pq}:

$$\mathbf{Q}=\begin{array}{c} \\ \\ p \\ \\ q \\ \\ \end{array}\begin{array}{c}\begin{array}{ccccc} & p & & q & \end{array}\\ \begin{pmatrix} \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot^{\cdot^{\cdot}} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & \cdots & Q_{pp} & \cdots & Q_{pq} & \cdots & \vdots \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ \vdots & \cdots & Q_{qp} & \cdots & Q_{qq} & \cdots & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ \cdot^{\cdot^{\cdot}} & \cdots & 0 & \cdots & 0 & \cdots & \ddots \end{pmatrix}\end{array}=\begin{pmatrix} \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdot^{\cdot^{\cdot}} \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ \vdots & \cdots & U_{pp} & \cdots & U_{pq} & \cdots & \vdots \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ \vdots & \cdots & U_{qp} & \cdots & U_{qq} & \cdots & \vdots \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ \cdot^{\cdot^{\cdot}} & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}\cdot\begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & s & \cdots & -c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix},$$

such that Q_{pq} becomes zero. For $Q_{pq}=0$ it must be required: $-U_{pp}c+U_{pq}s=0$. Hence, the expression for $\mathrm{tg}(\varphi_{pq})$ become $\mathrm{tg}(\varphi_{pq})=U_{pq}/U_{pp}$. This is equivalent to $(c,s)=\left(U_{pp}\Big/\sqrt{U_{pp}^2+U_{pq}^2},\ U_{pq}\Big/\sqrt{U_{pp}^2+U_{pp}^2}\right)$. For example,

$$\mathbf{Q}_N^{(1)}=\mathbf{U}_N\cdot\mathbf{J}(\varphi_{12})=\begin{pmatrix} , & + & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \end{pmatrix},\quad \mathbf{Q}_N^{(2)}=\mathbf{U}_N\cdot\mathbf{J}(\varphi_{12})\mathbf{J}(\varphi_{13})=\begin{pmatrix} , & + & + & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \end{pmatrix},$$

where white boxes are nonzero elements and black box is the zero element. Further,

$$\mathbf{Q}_N^{(2)}=\mathbf{U}_N\cdot\mathbf{J}(\varphi_{12})\mathbf{J}(\varphi_{13})\mathbf{J}(\varphi_{14})=\begin{pmatrix} , & + & + & + & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \end{pmatrix},\quad \ldots,\quad \mathbf{Q}_N^{(N-1)}=\mathbf{U}_N\cdot\mathbf{J}(\varphi_{12})\mathbf{J}(\varphi_{13})\cdots\mathbf{J}(\varphi_{1N})=\begin{pmatrix} , & + & + & + & + & + \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \\ , & , & , & , & , & , \end{pmatrix}.$$

But $\mathbf{Q}_N^{(N-1)}$ is an orthogonal matrix as the product of orthogonal matrices. For this reason it can have only the following form:

$$\mathbf{Q}_N^{(N-1)}=\mathbf{U}_N\cdot\mathbf{J}(\varphi_{12})\mathbf{J}(\varphi_{13})\ldots\mathbf{J}(\varphi_{1N})=\mathbf{U}_N\cdot\prod_{q=2}^{\to N}\mathbf{J}(\varphi_{1,q})=\left(\begin{array}{c|ccccc} \pm 1 & + & + & + & + & + \\ \hline + & , & , & , & , & , \\ + & , & , & , & , & , \\ + & , & , & \mathbf{Q}_{N-1} & , & , \\ + & , & , & , & , & , \\ + & , & , & , & , & , \end{array}\right),$$

where $\mathbf{Q}_{N-1}$ is $(N-1)\times(N-1)$ orthogonal matrix opposite to $\mathbf{Q}=\mathbf{Q}_N$ that is $(N\times N)$ orthogonal matrix. Obviously,

$$\mathbf{Q}_N^{\left(\frac{(N-1)N}{2}\right)}=\mathbf{U}_N\prod_{p=1}^{\to N-1}\prod_{q=p+1}^{\to N}\mathbf{J}(\varphi_{p,q})=\begin{pmatrix} \pm 1 & + & + & + & + & + \\ + & \pm 1 & + & + & + & + \\ + & + & \pm 1 & + & + & + \\ + & + & + & \pm 1 & + & + \\ + & + & + & + & \pm 1 & + \\ + & + & + & + & + & \pm 1 \end{pmatrix}.$$

Hence, an orthogonal matrix $\mathbf{U}$ is composed of series of Jacobi rotations: $\mathbf{U}(\boldsymbol{\varphi}) = \prod_{p=1}^{\leftarrow N-1} \prod_{q=p+1}^{\leftarrow N} \mathbf{J}(\varphi_{p,q})$, where $\boldsymbol{\varphi}=(\varphi_{1,2},\varphi_{1,3},...,\varphi_{N-1,N})$ is $N(N+1)/2$ -dimension vector of so-called the Jacobi angles φ_{pq}. Here $\prod_{i=0}^{\rightarrow n-1} \mathrm{T}^i = \mathrm{T}^0\mathrm{T}^1 \cdots \mathrm{T}^{n-1}$ and $\prod_{i=0}^{\leftarrow n-1} \mathrm{T}^i = \mathrm{T}^{n-1} \cdots \mathrm{T}^1\mathrm{T}^0$ are the right and left multiplications, respectively. Many-parameter representation $\mathbf{U}(\boldsymbol{\varphi}) = \prod_{p=1}^{\leftarrow N-1} \prod_{q=p+1}^{\leftarrow N} \mathbf{J}(\varphi_{p,q})$ is very important with theoretical point of view, but it is not very useful with digital processing point of view.

The concept of fast MPT in signal and image processing was printed by Andrews [4] in the form of tensor product of Jacobi (2×2)-matrices $\mathbf{J}_2(\varphi_i) = \begin{bmatrix} \cos\varphi_i & \sin\varphi_i \\ \sin\varphi_i & -\cos\varphi_i \end{bmatrix}$, $i = 1,2,...,n$:

$$\mathbf{CS}_{2^n}(\varphi_1,\varphi_2,...,\varphi_n) = \mathbf{J}_2(\varphi_1)\otimes\cdots\otimes\mathbf{J}_2(\varphi_n) = \begin{bmatrix} \cos\varphi_1 & \sin\varphi_1 \\ \sin\varphi_1 & -\cos\varphi_1 \end{bmatrix}\otimes\cdots\otimes\begin{bmatrix} \cos\varphi_n & \sin\varphi_n \\ \sin\varphi_n & -\cos\varphi_n \end{bmatrix}.$$

This tensor product is factorized into the ordinary product of sparse matrices

$$\mathbf{CS}_{2^n}(\varphi_1,\varphi_2,...,\varphi_n) = \prod_{i=1}^{n} \left[I_{2^{n-i}} \otimes \mathbf{J}_2(\varphi_i) \otimes I_{2^{i-1}} \right].$$

It is just the fast Andrews transform (FAT). In particular case, when $\varphi_1 = \varphi_2 = ... = \varphi_n \equiv \pi/4$, we obtain ordinary Walsh transform $\mathbf{W}_{2^n} = \left(\sqrt{2}/2\right)^n \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\otimes\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\otimes\cdots\otimes\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Analogously form has n-parameter Haar transform [7-10]

$$\mathbf{HT}_{2^n}(\varphi_1,\varphi_2,...,\varphi_n) = \prod_{i=1}^{n}\left[\left(\mathbf{J}_2(\varphi_i)\otimes I_{2^{n-i}}\right) \oplus I_{2^n-2^{n-i+1}} \right] P_{2^n},$$

where P_{2^n} is the perfect shuffle permutation matrix [11]. Obviously,

$$\mathbf{HT}_{2^n}\left(\frac{\pi}{4},\frac{\pi}{4},...,\frac{\pi}{4}\right) \equiv \mathbf{HT}_{2^n} = \prod_{i=1}^{n}[(\mathbf{W}_2\otimes I_{2^{n-i}})\oplus I_{2^n-2^{n-i+1}}]P_{2^n}$$

is the ordinary Haar transform. Recently, several authors [10]-[20] have proposed Jacobi parametrization of wavelet transforms.

In the first part of our work [12], we proposed anti-eavesdropping and anti-jamming Intelligent OFDM telecommunication system (TCS), based on many-parameter transforms (MPTs). In this part, we propose two novel families of MPTs: 1) many-parameter wavelet transform (MPWT) $\mathbf{WT}_{2^n}(\varphi_1,\varphi_2,...,\varphi_q)$ for Intelligent MPWT-OFDM-TCS and 2) many-parameter Fourier transform (MPFT) $\mathrm{F}^{(\alpha_0,\alpha_1,...,a_{N-1})}$ for Intelligent MPFT-OFDM-TCS.

The contents of the paper are organized as follows. Section 2 of the paper presents a brief introduction to the novel many-parameter wavelet transforms and packets (MPWP). Sections 3 presents many-parameter and fractional Fourier transforms.

2. Many-parameter wavelet transforms

The main goal of this section is to show that wavelet transforms and packets have the multi-parametric representations in the form of a product of the Jacobi rotation matrices. The wide class of orthogonal cyclic wavelet transforms $\mathbf{WT}$ can be defined by two sets of coefficients: $h_0, h_1, \ldots, h_{L-1}$ and $g_0, g_1, \ldots, g_{L-1}$, where $L = 2D$ is an even number [13]-[14]. In fact $\mathbf{WT}$ is determined only by a set of h-coefficients h_0, h_1, …, h_{L-1} since the second set of coefficients is usually assigned according to the rule $g_0 = h_{L-1}$, $g_1 = -h_{L-2}$, …, $g_{L-1} = -h_0$. For this reason, we will designate wavelet transform as

$\mathbf{WT}_{2^n}\left[h_0,h_1,\ldots,h_{L-1}\right]$. Let $m=\left]\log_2 D\right[$ be the smallest positive integer such that the wavelet transform $\mathbf{WT}_{2^n}$ is factorized into a product of sparse matrixes, named stairs-like *atomic wavelet transforms* $\mathbf{AWT}_{2^{r+1}}\left[h_0,h_1,\ldots,h_{L-1}\right]$:

$$\mathbf{WT}_{2^n}[h_0,h_1,...,h_{2D-1}]=\prod_{r=m}^{\rightarrow n-1}\left[\mathbf{AWT}_{2^{r+1}}[h_0,h_1,...,h_{2D-1}]\oplus\mathbf{I}_{2^n-2^{r+1}}\right]. \tag{2}$$

For example, for the (8×8)-Daubechies-4 wavelet transform we have

$$\mathbf{WT}_8\left[h_0,h_1,h_2,h_3\right]=\left[\mathbf{AWT}_4\oplus\mathbf{I}_4\right]\cdot\left[\mathbf{AWT}_8\right]=\left[\begin{pmatrix} h_0 & h_1 & h_2 & h_3\\ h_2 & h_3 & h_0 & h_1\\ \hline g_0 & g_1 & g_2 & g_3\\ g_2 & g_3 & g_0 & g_1\end{pmatrix}\oplus\mathbf{I}_4\right]\cdot\begin{pmatrix} h_0 & h_1 & h_2 & h_3 & & & & \\ & & h_0 & h_1 & h_2 & h_3 & & \\ & & & & h_0 & h_1 & h_2 & h_3\\ h_2 & h_3 & & & & & h_0 & h_1\\ \hline g_0 & g_1 & g_2 & g_3 & & & & \\ & & g_0 & g_1 & g_2 & g_3 & & \\ & & & & g_0 & g_1 & g_2 & g_3\\ g_2 & g_3 & & & & & g_0 & g_1\end{pmatrix}.$$

We are going to prove that multi-parametric representation of wavelet transform exists and that it depends on D angle-parameters. In order to find multi-parametric form of wavelet transform we use the rotation-reflection Jacobi $(2^n\times 2^n)$-matrix $\mathbf{J}_{p,q}(\varphi)$. For this matrix we have $\mathbf{J}_{p,q}(\varphi)\mathbf{J}_{p,q}(\varphi)=\mathbf{I}$ and $\mathbf{J}_{p,q}^{-1}(\varphi)=\mathbf{J}_{p,q}(\varphi)$. We are going to multiply the atomic wavelet matrix $\mathbf{AWT}_{2^{r+1}}$ by $\mathbf{J}_{p,q}(\varphi)$-matrices sequentially with such choice of angles $\varphi_k,...,\varphi_1,\varphi_0$ for any k, that product $\mathbf{J}_{p_k,q_k}(\varphi_k)\cdot\ldots\cdot\mathbf{J}_{p_1,q_1}(\varphi_1)\cdot\mathbf{J}_{p_0,q_0}(\varphi_0)\cdot\mathbf{AWT}_{2^{r+1}}\left[h_0,h_1,\ldots,h_{L-1}\right]$ will be a permutation matrix $\mathbf{P}_{2^{r+1}}$ or unit matrix $\mathbf{I}_{2^{r+1}}$. It gives the following result:

$$\mathbf{AWT}_{2^{r+1}}\left[h_0,h_1,\ldots,h_{2D-1}\right]=\mathbf{AWT}_{2^{r+1}}\left[\varphi_0,\varphi_1,\ldots,\varphi_{D-1}\right]\mathbf{P}_{2^{r+1}}=\left(\prod_{i=0}^{\rightarrow D-1}\prod_{k=0}^{2^r-1}\mathbf{J}_{i\underset{2^r}{\oplus}k,2^r+k}(\varphi_i)\right)\mathbf{P}_{2^{r+1}}, \tag{3}$$

where $\underset{2^r}{\oplus}$ is addition modulo 2^r. The classical cyclic wavelet transform is constructed from atomic wavelet transforms according to (2). Substituting (10) in (2) we obtain the multi-parametric presentation of wavelet transforms

$$\mathbf{WT}_{2^n}[h_0,h_1,...,h_{2D-1}]=\mathbf{WT}_{2^n}[\varphi_0,\varphi_1,...,\varphi_{D-1}]=\prod_{r=m}^{\rightarrow n-1}\left[\left(\prod_{i=0}^{\rightarrow D-1}\prod_{k=0}^{2^r-1}\mathbf{J}_{i\underset{2^r}{\oplus}k,2^r+k}(\varphi_i)\right)\mathbf{P}_{2^{r+1}}\oplus\mathbf{I}_{2^n-2^{r+1}}\right]. \tag{4}$$

MPT $\mathbf{WT}_{2^n}[\varphi_0,\varphi_1,...,\varphi_{D-1}]$ has the form of the product of the sparse Jacoby matrixes, which describes a fast algorithm for this transform. It is possible to obtain all the transforms of $\mathbf{WT}_{16}\left[h_0,h_1,h_2,h_3\right]$-type by changing the angles φ_0 and φ_1. In Fig. 1 we see wavelet function (mother and farther functions) as subcarriers for different values of parameters φ_0 and φ_1.

All the atomic matrices in multi-parametric representation of wavelet transform are characterized by the same set of angle-parameters. All angles have equal values in each atomic matrix and have to be chosen synchronously. Of course, it is possible to use different angle sets in different atomic matrixes and to change them not synchronously. In this case we will get heterogeneous many-parameter wavelet transforms. For example, let us introduce

$$\mathbf{AWT}_{2^{r+1}}\left[\boldsymbol{\varphi}_0^r,\boldsymbol{\varphi}_1^r,...,\boldsymbol{\varphi}_{D-1}^r\right]=\left(\prod_{i=0}^{\rightarrow D-1}\prod_{k=0}^{2^r-1}\mathbf{J}_{i\underset{2^r}{\oplus}k,2^r+k}(\varphi_{i,k}^r)\right)\mathbf{P}_{2^{r+1}}, \tag{5}$$

where $\boldsymbol{\varphi}_0^r := (\varphi_{0,0}^r, \varphi_{0,1}^r, \ldots, \varphi_{0,2^r-1}^r)$, $\boldsymbol{\varphi}_1^r := (\varphi_{1,0}^1, \varphi_{1,1}^1, \ldots, \varphi_{1,2^r-1}^1), \ldots, \boldsymbol{\varphi}_{D-1}^r := (\varphi_{D-1,0}^r, \varphi_{D-1,1}^r, \ldots, \varphi_{D-1,2^r-1}^r)$. In this case $\mathbf{AWT}_{2^{r+1}}$ depend on $D\cdot 2^r$ parameters. A heterogeneous many-parameter wavelet transforms will have the following form

$$\mathbf{WT}_{2^n}[\boldsymbol{\varphi}_0^m, \boldsymbol{\varphi}_1^m, \ldots, \boldsymbol{\varphi}_{D-1}^m;\ \boldsymbol{\varphi}_0^{m+1}, \boldsymbol{\varphi}_1^{m+1}, \ldots, \boldsymbol{\varphi}_{D-1}^{m+1};\ \ldots,\ \boldsymbol{\varphi}_0^n, \boldsymbol{\varphi}_1^n, \ldots, \boldsymbol{\varphi}_{D-1}^n] = \overset{\rightarrow n-1}{\prod_{r=m}} \left[\mathbf{AWT}_{2^{r+1}}[\boldsymbol{\varphi}_0^r, \boldsymbol{\varphi}_1^r, \ldots, \boldsymbol{\varphi}_{D-1}^r]\cdot \mathbf{P}_{2^{r+1}} \oplus \mathbf{I}_{2^n - 2^{r+1}} \right] =$$

$$= \overset{\rightarrow n-1}{\prod_{r=m}} \left[\left(\overset{\rightarrow D-1}{\prod_{i=0}} \prod_{k=0}^{2^r-1} \mathbf{J}_{\substack{i\oplus k, 2^r+k \\ 2^r}} (\varphi_{i,k}^r) \right) \mathbf{P}_{2^{r+1}} \oplus \mathbf{I}_{2^n - 2^{r+1}} \right].$$

It depend on $q = D\cdot 2^m + D\cdot 2^{m+1} + \ldots + D\cdot 2^{n-1} = 2^m \cdot D\cdot\left(2^{n-m} - 1\right)$ parameters.

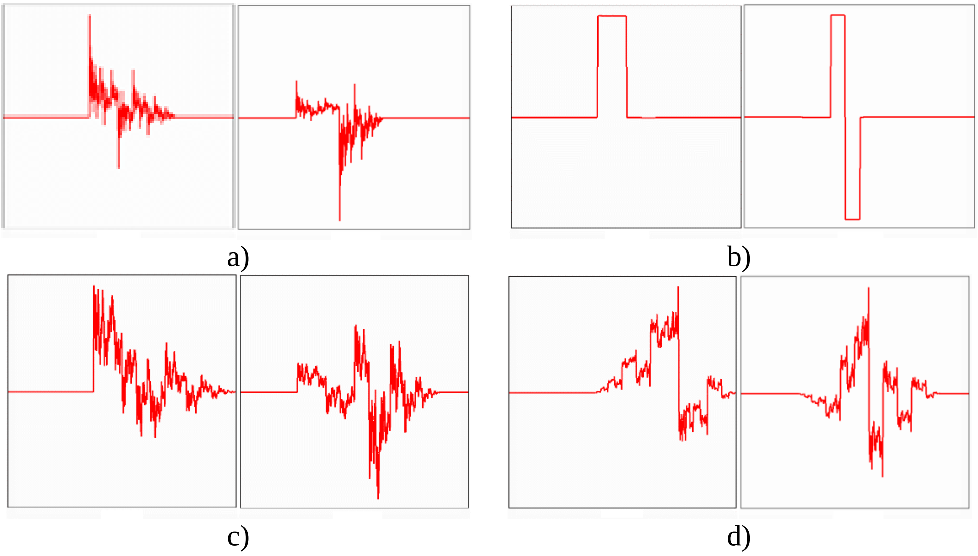

Figure 1. Wavelet mother and farther functions as subcarriers for different values of parameters φ_0 and φ_1: a) $\varphi_0 = 204^0, \varphi_1 = 291^0$, b) $\varphi_0 = 180^0, \varphi_1 = 135^0$, c) $\varphi_0 = 199^0, \varphi_1 = 80^0$, d) $\varphi_0 = 138^0, \varphi_1 = 337^0$.

3. Fractional and many-parameter Fourier transforms

The eigen–decomposition (ED) is a tool of both practical and theoretical importance in digital signal and image processing. The ED transforms are defined by the following way. Let U be an arbitrary discrete orthogonal (or unitary) $(N\times N)$–transform, λ_k and $\left|\Psi_m(n)\right\rangle$, $m,n = 0,1,\ldots,N-1$, be its eigen–values and eigen column-vectors, respectively. Let $\mathbf{U} = \left[\left|\Psi_0(n)\right\rangle, \left|\Psi_1(n)\right\rangle, \ldots, \left|\Psi_{N-1}(n)\right\rangle\right]$ be the matrix of eigen-vectors of the U–transform. Then $\mathbf{U}^{-1}\cdot \mathrm{U}\cdot \mathbf{U} = \mathbf{Diag}\left\{\lambda_0, \ldots, \lambda_{N\text{-}1}\right\} = \Lambda$. Hence, we have the following eigen–decomposition: $\mathrm{U} = \left[u_k(n)\right] := \sum_{m=0}^{N-1} \lambda_m \left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right| = \mathbf{U}\cdot \mathbf{Diag}\left(\lambda_0, \ldots, \lambda_{N\text{-}1}\right)\cdot \mathbf{U}^{-1}$.

Definition 1. *For an arbitrary real numbers $a_0, \ldots, a_{N-1}$ we introduce the multi-parameter U–transform*

$$\mathrm{U}^{(a_0,\ldots,a_{N-1})} := \mathbf{U}\cdot \mathbf{Diag}\left(\lambda_0^{a_0}, \ldots, \lambda_{N\text{-}1}^{a_{N\text{-}1}}\right)\cdot \mathbf{U}^{-1}. \tag{6}$$

If $a_0 = \ldots = a_{N-1} \equiv a$ then this transform is called the fractional U–transform. For this transform we have

$$\mathrm{U}^a := \mathbf{U}\left\{\mathrm{diag}\left(\lambda_0^a, \ldots, \lambda_{N\text{-}1}^a\right)\right\}\mathbf{U}^{-1} = \mathbf{U}\,\Lambda^a\,\mathbf{U}^{-1}. \tag{7}$$

The zeroth-order fractional U–transform is equal to the identity transform: $\mathrm{U}^0 = \mathbf{U}\Lambda^0\mathbf{U}^{-1} = \mathbf{U}\mathbf{U}^{-1} = \mathbf{I}$ and the first-order fractional U-transform operator is equal to the initial transform $\mathrm{U}^1 = \mathbf{U}\Lambda\mathbf{U}^{-1}$.
The families $\left\{\mathrm{U}^{(\alpha_0,\ldots,\alpha_{N-1})}\right\}_{(\alpha_0,\ldots,\alpha_{N-1})\in\mathbf{R}^N}$ and $\left\{\mathrm{U}^a\right\}_{a\in\mathbf{R}}$ form multi- and one-parameter continuous unitary groups with multiplications $\mathrm{U}^{(a_0,\ldots,a_{N-1})}\mathrm{U}^{(b_0,\ldots,b_{N-1})} = \mathrm{U}^{(a_0+b_0,\ldots,a_{N-1}+b_{N-1})}$ and $\mathrm{U}^a\mathrm{U}^b = \mathrm{U}^{a+b}$.
Let $\mathrm{F}_N = \left[e^{-j\frac{2\pi}{N}kn}\right]_{k,n=0}^{N-1}$ be the discrete Fourier $(N\times N)$–transform (DFT). Relevant properties are that the square $\left(\mathrm{F}_N^2 f\right)(x) = f(-x)$ is the inversion operator, and that its fourth power $\left(\mathrm{F}_N^4 f\right)(x) = f(x)$ is the identity; hence $\mathrm{F}_N^3 = \mathrm{F}_N^{-1}$ The operator F_N thus generates the Fourier cyclic group of order 4: $\mathbf{Gr}_4(\mathrm{F}) = \left\{\mathrm{F}_N^a\right\}_{a\in\{0,1,2,3\}} = \left\{I, \mathrm{F}_N^1, \mathrm{F}_N^2, \mathrm{F}_N^3\right\}$.
The idea of fractional powers of the Fourier operator F appears in the mathematical literature [21-28]. This idea is to consider the eigen-value decomposition of the Fourier transform $\mathrm{F} \equiv \sum_{n=0}^{\infty}\lambda_n\left|\Psi_n(x)\right\rangle\left\langle\Psi_n(\omega)\right|$ in terms of eigen-values $\lambda_n = e^{jn\pi/2} = j^n$ and eigen-functions $\Psi_n(x)$ in the form of the Hermite functions. The family of FrFT $\left\{\mathrm{F}^a\right\}_{a\in[0,4)}$ (instead of $\left\{\mathrm{F}^a\right\}_{a\in\{0,1,2,3\}}$) is constructed by replacing the n-th eigen-value $\lambda_n = e^{jn\pi/2}$ by its a-th power $\lambda_n^a = e^{jn\pi a/2}$, for a between 0 and 4.
The eigenvalues of the standard DFT matrix F_N are the fourth roots of unity, to be denoted by $\lambda_s \in \left\{e^{j\pi s/2}\right\}_{s=0}^{3} \in \{\pm1, \pm j\}$ and $\left\{\Psi_m(n)\right\}_{m=0}^{N-1}$ are the discrete Hermite polynomials. This divides the space of *N*-point complex signals into four Fourier invariant subspaces whose dimensions N_s are the multiplicities of the eigenvalues λ_s, which have a modulo 4 recurrence in the dimension $N = 2^N = 4M$ given by $N_0 = M+1, N_1 = M-1,$ $N_2 = M,\ N_3 = M.$ Let $s(n): \{0,1,2,\ldots,\mathrm{N}-1\} \to \{0,1,2,3\}$ be a peculiar function. It determines a distribution of eigen-values along main diagonal $\mathbf{Diag}\left(e^{j\frac{\pi}{2}s(n)a}\right)$ in (8). This function takes $M+1$ times value 0, $M-1$ times value 1, and M times values 2 and 3. In particular, $s(0)=0$.

Definition 2. The discrete classical and Bargmann fractional Fourier transforms are defined as

$$\mathrm{F}^a = \left[e_k^{(a)}(n)\right] := \mathbf{U}\left\{\mathbf{Diag}\left(e^{j\frac{\pi}{2}s(m)a}\right)\right\}\mathbf{U}^{-1} = \sum_{m=0}^{N-1} e^{j\frac{\pi}{2}s(m)a}\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|, \tag{8}$$

$$\mathrm{BF}^a = \left[be_k^{(a)}(n)\right] := \mathbf{U}\left\{\mathbf{Diag}\left(e^{j\frac{\pi}{2}ma}\right)\right\}\mathbf{U}^{-1} = \sum_{m=0}^{N-1} e^{j\frac{\pi}{2}ma}\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|, \tag{9}$$

Definition 3. The discrete classical-like and Bargmann-like multi-parameter DFT we define by the following way

$$\mathrm{F}^{(\mathbf{a})} = \mathrm{F}^{(a_0,a_1,a_2,\ldots,a_{N-1})} = \left[e_k^{(\mathbf{a})}(n)\right] = \mathbf{U}\left\{\mathrm{diag}\left(e^{j\frac{\pi}{2}s(m)a_m}\right)\right\}\mathbf{U}^{-1} = \sum_{m=0}^{N-1} e^{j\frac{\pi}{2}s(m)a_m}\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|, \tag{10}$$

$$\mathrm{BF}^{(\mathbf{a})} = \mathrm{BF}^{(a_0,a_1,a_2,\ldots,a_{N-1})} = \left[be_k^{(\mathbf{a})}(n)\right] := \mathbf{U}\left\{\mathrm{diag}\left(e^{j\frac{\pi}{2}ma_m}\right)\right\}\mathbf{U}^{-1} = \sum_{m=0}^{N-1} e^{j\frac{\pi}{2}ma_m}\left|\Psi_m(k)\right\rangle\left\langle\Psi_m(n)\right|, \tag{11}$$

where $\mathbf{a} = (a_0, a_1, a_2, \ldots, a_{N-1})$.

The parameters $(a_1,\ldots,a_{N-1})$ and a can have any real values. For each fixed values $(a_1^*,\ldots,a_{N-1}^*)$ and a^* we obtain concrete transforms $\mathcal{F}^{(a_1^*,\ldots,a_{N-1}^*)}$ and $\mathcal{F}^{a^*}$ which are called the realizations of MPFT $\mathcal{F}^{(a_1,\ldots,a_{N-1})}$ and FrFT $\mathcal{F}^{a}$, respectively. All realizations of $\mathcal{F}^{(a_1,\ldots,a_{N-1})}$ and $\mathcal{F}^{a}$ form two ensembles of transforms.The operators $\mathcal{F}^{(a_1,\ldots,a_{N-1})}$ and $\mathcal{F}^{a}$ are periodic in each parameter with period 4 since $\mathcal{F}^{4}=I$ and hence $\mathcal{F}^{(a_1,\ldots,a_{N-1})}\mathcal{F}^{(b_1,\ldots,b_{N-1})}=\mathcal{F}^{\left(a_1\underset{4}{\oplus}b_1,\ldots,a_{N-1}\underset{4}{\oplus}b_{N-1}\right)}$ and $\mathcal{F}^{a}\mathcal{F}^{b}=\mathcal{F}^{\left(a\underset{4}{\oplus}b\right)}$, where $a_i\underset{4}{\oplus}b_i=(a_i+b_i)\operatorname{mod}4,\ \forall i=1,...,N-1$. Consequently, the ranges of $(a_1,\ldots,a_{N-1})$ and a are tori $\mathbf{Tor}_4^{N-1}=(Z/4Z)^{N-1}=[0,4)^{N-1}$ and $\mathbf{Tor}_4^1:=Z/4Z\equiv[0,4)$.

In the case of α–parameterization $\mathcal{F}^{(\boldsymbol{\theta})}=\mathcal{F}^{(\alpha_0,\alpha_1,\ldots,\alpha_{N-1})}$ and $\mathcal{F}^{\alpha}$, where $\boldsymbol{\theta}=(\alpha_0,\alpha_1,\ldots,\alpha_{N-1})$, we have $\alpha_i\underset{2\pi}{\oplus}\beta_i=(\alpha_i+\beta_i)\operatorname{mod}2\pi,\ \forall i=0,1,...,N-1$. Consequently, the ranges of $(\alpha_0,\ldots,\alpha_{N-1})$ and α are tori $\mathbf{Tor}_{2\pi}^{N-1}=(Z/2\pi Z)^{N-1}=[0,2\pi)^{N-1}$ and $\mathbf{Tor}_{2\pi}^1:=Z/2\pi Z=[0,2\pi)$, respectively. Hence, ensembles

$$\left\{\mathcal{F}^{(a_0,a_1,\ldots,a_{N-1})}\middle|(a_0,a_1,\ldots,a_{N-1})\in\mathbf{Tor}_4^{N-1}\right\}\text{ and }\left\{\mathcal{F}^{a}\middle|a\in\mathbf{Tor}_4\right\}\tag{12}$$

form two multiplicative groups are isomorphic to $(Z/4Z)^{N-1}$ and $Z/4Z$ in a–parametrization, respectively. In α–parametrization they form two multiplicative groups

$$\left\{\mathcal{F}^{(\boldsymbol{\theta})}=\mathcal{F}^{(\alpha_0,\alpha_1,\ldots,\alpha_{N-1})}\middle|\boldsymbol{\theta}=(\alpha_0,\alpha_1,\ldots,\alpha_{N-1})\in\mathbf{Tor}_{2\pi}^{N-1}\right\}\text{ and }\left\{\mathcal{F}^{\alpha}\middle|\alpha\in\mathbf{Tor}_{2\pi}\right\}.\tag{13}$$

In (12) and (13) $a_0=\alpha_0\equiv0$.

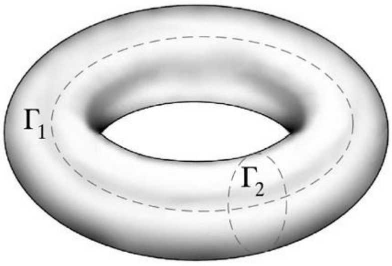

Figure 2. The two topological independent curves Γ_1 and Γ_2 on a two-dimensional torus.

Any $(N-1)$-D torus $\mathbf{Tor}_{2\pi}^{N-1}$ is a periodic object which can be considered as the product of $N-1$ independent periodicities: $\mathbf{Tor}_{2\pi}^{N-1}=\underbrace{\mathbf{Tor}_{2\pi}^1\times\mathbf{Tor}_{2\pi}^1\times\ldots\times\mathbf{Tor}_{2\pi}^1}_{N-1\text{ times}}$. In other words, we can define $N-1$ topologically independent closed curves, $\Gamma_1,\Gamma_2,...,\Gamma_{N-1}$, on a given torus, where none of the Γ_i can be deformed continuously into each other or shrunk to zero. In Fig. 2 we represent a 2-D torus for which Γ_1 turns around through the longest path while Γ_2 does it through the shortest path. Note that neither Γ_1 nor Γ_2 can be converted into each other by continuous transformations. In effect, let us denote by $\Delta\alpha_i$ the change of the angle variable α_i. If angle variable α_i changes in 2π, then the $\mathcal{F}^{(a_1,\ldots,a_{N-1})}$ executes a complete oscillation along the curve Γ_i and no change otherwise.

Now we are going to show, that a many-parameter Fourier transform has 1-parameter representation.

Definition 4. Let $\mathbf{l}_1=(1,0,...,0),\mathbf{l}_2=(0,1,...,0),...,\mathbf{l}_{N-1}=(0,0,...,1)$ be a finite set of vectors $\mathbf{l}_1,\mathbf{l}_2,...,\mathbf{l}_{N-1}\in\mathbf{Tor}_{2\pi}^{N-1}$, define $\boldsymbol{\omega}=(\omega_1,\omega_2,...,\omega_{N-1})=\sum_{i=1}^{N-1}\omega_i\mathbf{l}_i$, where $\omega_i=2\pi f_i\geq0$. Then the set

$t\boldsymbol{\omega} = (t\omega_1, t\omega_2, ..., t\omega_{N-1}) = \sum_{i=1}^{N-1} t\omega_i \mathbf{l}_i$ is called the trajectory on $\mathbf{Tor}_{2\pi}^{N-1}$ along the frequency vector $\boldsymbol{\omega} = (\omega_1, \omega_2, ..., \omega_{N-1})$. If $(\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) = (t\omega_1, t\omega_2, ..., t\omega_{N-1})$ then

$$\mathrm{F}^{(\alpha_0,\alpha_1,\ldots,\alpha_{N-1})} = \mathrm{F}^{(t\omega_0, t\omega_1, t\omega_2, ..., t\omega_{N-1})} = \mathrm{F}^{t(\omega_0,\omega_1,\omega_2,...,\omega_{N-1})} = \mathrm{F}^{t\boldsymbol{\omega}} = \left(\mathrm{F}^{\boldsymbol{\omega}}\right)^t$$

is multiply periodic operator-valued functions with $N-1$ independent (angular) frequencies $\omega_1, \omega_2, ..., \omega_{N-1}$, and $\omega_0 = 0$. However, this property does not imply that, in general, the $\mathrm{F}_N^{(\alpha_0,\ldots,\alpha_{N-1})}$ is (simply) periodic functions; for it would be necessary that there exists *a single period* Ω_0 for which $\mathrm{F}^{t((\omega_1+\Omega_0),(\omega_2+\Omega_0),...,(\omega_{N-1}+\Omega_0))} = \mathrm{F}^{t(\omega_1,\omega_2,...,\omega_{N-1})}$ is periodic. This is the case if, and only if, the frequencies $(\omega_1, \omega_2, ..., \omega_{N-1})$ are integer multiples of a single frequency Ω_0: $\omega_i = p_j \Omega_0$, $i = 1, 2, ..., N-1$, where $p_j = 0, \pm 1, \pm 2, ..$ are integer numbers. Equation $\omega_i = p_j \Omega_0$ means that in order to have periodic motion, the frequencies must be *commensurable*. From $\omega_i = p_j \Omega_0$ we immediately see that this is equivalent to assuming that all frequencies are rational multiples of each other: $\frac{\omega_i}{\omega_j} = \frac{p_i}{p_j} =$ a rational number. If the frequencies are *incommensurable*, in other words, if they are not rationally related, then the motion is termed multiply periodic or quasiperiodic or conditionally periodic, according to different terminologies in use, and never repeats itself.

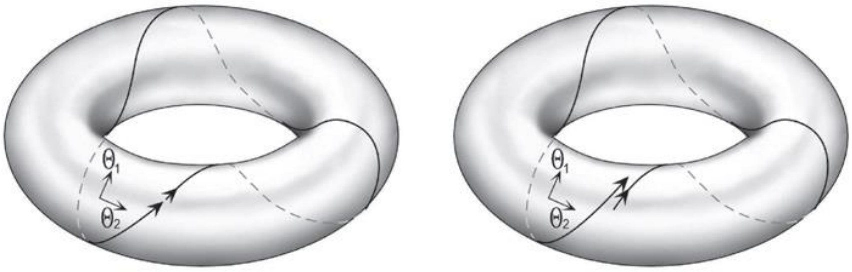

Figure 3. The Comparison of trajectories on 2-D tori. The curve in (a) is a rational trajectory, where $\omega_1 / \omega_2 = 3,$, that is, the trajectory closes over itself after three turns around Γ_1 and one turn around Γ_2. The curve in (b) is an irrational trajectory, where the frequencies are not commensurable. In (b) the trajectory will eventually cover the surface of the torus densely.

We can, therefore, conclude that on a given torus, the trajectory will be a closed curve (i.e. the motion of the system will be periodic) if and only if the frequencies of the motion are commensurable. When frequencies are incommensurable the trajectory will densely cover the torus, never closing on itself. In the first case, we call it rational, or resonant, trajectory, while in the latter irrational, or nonresonant, trajectory (see Fig. 3). For this reason, a many-parameter Fourier transform $\mathrm{F}^{(\alpha_0,\alpha_1,\ldots,a_{N-1})} = \mathrm{F}^{t(\omega_0,\omega_1,...,\omega_{N-1})} = \mathrm{F}^{t\boldsymbol{\omega}} = \left(\mathrm{F}^{\boldsymbol{\omega}}\right)^t$ is an one-parameter periodic representation of MPFT $\mathrm{F}^{(\alpha_0,\alpha_1,\ldots,a_{N-1})}$ if trajectory $t\boldsymbol{\omega}$ is resonant, and MPFT is an one-parameter quasi-periodic representation of MPFT $\mathrm{F}^{(\alpha_0,\alpha_1,\ldots,a_{N-1})}$ if trajectory $t\boldsymbol{\omega}$ is nonresonant. In both cases MPFT is fractional power of the transform $\mathrm{F}^{\boldsymbol{\omega}}$ but not F.

Let us introduce the uniformly discretization (sampling) of t-parameter: $t^{(k)} = k\Delta t$ $(k = 0, 1, ..., M-1)$. We have discrete values $\alpha_i^k = t^{(k)}\omega_i = k\Delta t\omega_i$ and obtain transform with single discrete parameter $\mathrm{F}^{(\alpha_0,\alpha_1,\ldots,\alpha_{N-1})} \xrightarrow{Discr} \mathrm{F}^{k(\Delta t\omega_0, \Delta t\omega_1, \Delta t\omega_2, ..., \Delta t\omega_{N-1})}$. Discretization (sampling) of parameter t give discrete trajectory on torus (see Fig. 4).

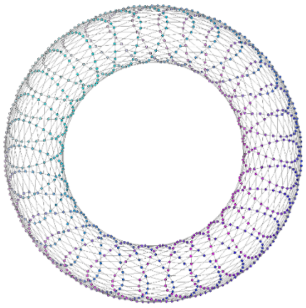

Figure 4. Discrete trajectory.

4. Conclusion

In this paper, we have shown a new unified approach to the many-parametric representation of orthogonal wavelet and Fourier transforms. This form is the product of sparse rotation matrixes and it describes fast algorithms for introduced many-parameter transforms. Defined representation of many-parameter transforms (MPT) depend on finite set of free parameters, which could be changed independently of one another. For each set of values of parameter we get the unique orthogonal transform. We are going to use these MPTs for constructing of two novel Intelligent OFDM-telecommunication systems The new systems will use inverse MPT (or inverse MPT) for modulation at the transmitter and direct MPT (or direct MPT) for demodulation at the receiver.

5. References

[1] Jacobi, C.G.J. Uber ein leichtes verfahren die in der theorie der sacularstorungen vorkommendern gleichungen. numerische aufzulosen // Jurnal fur die reine und angewandte Mathematik. – 1846. – Vol. 30. – P. 51-94.

[2] Brent, R.P. The solution of singular-value and symmetric eigenvalue problems on multiprocessor Arrays / R.P. Brent, F.T. Luk // SIAM Journal on Scientific and Statistical Computing. – 1985. – Vol. 6(1). – P. 69-83.

[3] Modi, J.J. Efficient implementation of Jacobi's diagonalization method on the DAP / J.J. Modi, J.D. Pryce // Numerische Mathematik. – 1985. – Vol. 46. – P. 443-454.

[4] Andrews, H.C. Computer Techniques in Image Processing. – New York: Academic Press, 1970. – 244 p.

[5] Labunets, V.G. Multi-parameter Golay 2-complementary sequences and transforms / V.G. Labunets, V.P. Chasovskikh, E. Ostheimer // Proceedings of the 4th International Youth Conference on Information technologies and nanotechnologies. – 2018. – P. 1013-1022.

[6] Labunets, V.G. Multiparameter Golay *m*-complementary sequences and transforms / V.G. Labunets, V.P. Chasovskikh, E. Ostheimer // Proceedings of the 4th International Youth Conference on Information technologies and nanotechnologies. – 2018. – P. 1005-1012.

[7] Labunets, V.G. Generalized Haar transforms. Topics in Multi–valued elements, structures and systems. – Kiev: Institute of Cybernetics of Ukraian Academy of Sciences Press, 1983. – P. 46-58.

[8] Labunets, V.G. Unified approach to fast transform algorithm // Topics in Orthogonal methods in signal processing and system analysis. – Sverdlovsk: Urals Polytechnical Institute Press, 1980. – P. 4-14.

[9] Labunets, V.G. Fast multiparameter transforms // Proceedings of Radioelectronics. – 1985. – Vol. 8. – P. 89-109.

[10] Labunets, V.G. Unified approach to fast algorithms of unitary transforms (in Russian) // Topics in Multi–valued elements, structures and systems. – Kiev: Institute of Cybernetics of Ukraian Academy of Sciences Press, 1983. – P. 58-70.

[11] Vedral, V.V. Quantum networks for elementary arithmetic operations / V.V. Vedral, A. Barenco, A. Ekert // Physical Review. – 1996. – Vol. A54. – P. 147-153.

[12] Labunets, V. Intelligent OFDM telecommunication system. Part 1. Model of system / V. Labunets, E. Ostheimer // Presented at this Proceedings.

[13] Daubechies, I. Factoring wavelet transforms into lifting steps / I. Daubechies, W. Sweldens // J. Fourier Anal. Appl. – 1998. – Vol. 4(3). – P. 247-269.
[14] Daubechies, I. Ten lectures on wavelets // Society for Industrial and Applied Mathematics. – Philadelphia: PA, 1992. – 68 p.
[15] Labunets, V. Many-parametric cyclic wavelet transforms. Part 1. The first and second canonical forms / V. Labunets, K. Egiazarian, J. Astola, E. Ostheimer // Proceedings of the International TICSP Workshop on Spectral Methods and Multirate Signal Processing, 2007. – P. 111-120.
[16] Labunets, V.G. Many-parametric cyclic wavelet transforms. Part 2. The third and fourth canonical forms / V. Labunets, K. Egiazarian, J. Astola, E. Ostheimer // Proceedings of the International TICSP Workshop on Spectral Methods and Multirate Signal Processing, 2007. – P. 121-132.
[17] Labunets, V.G. Fast multi-parametric wavelet transforms and packets for image processing / V.G. Labunets, D.E. Komarov, E.V. Ostheimer // CEUR Workshop Proceedings. – 2016. – Vol. 1710. – P. 134-145.
[18] Labunets, V. Multi-parametric wavelet transforms and packets / V. Labunets, D. Gainanov, D. Berenov // Proceedings of the 11th International Conference on Pattern Recognition and Image Analysis: New Information Technologies. – 2013. – Vol. 1. – P. 52-56.
[19] Labunets, V. The best multi-parameter wavelet transforms / V. Labunets, D. Gainanov, D. Berenov // Proceedings of the 11th International Conference on Pattern Recognition and Image Analysis: New Information Technologies. – 2013. – Vol. 1. – P. 56-60.
[20] Lebedev, S.A. Cyclic many-parametric Wavelet Transforms: How Many are there Optimal (with image compression point of view) Wavelet Transforms? / S.A. Lebedev, V.G. Labunets, O.A. Gusev, E.V. Ostheimer-Labunets // Proceedings of the 8th International Conference on Digital Signal Processing and Its Applications, 2006. – P. 113-116.
[21] Condon, E.U. Immersion of the Fourier transform in a continuous group of functional transforms / E.U. Condon // Proc. Nat. Acad. Sci. – 1937. – Vol. 23. – P. 158-164.
[22] Kober, H. Wurzeln aus der Hankel- und Fourier und anderen stetigen Transformationen / H. Kober // Quart. J. Math. Oxford Ser. – 1939. – Vol. 10. – P. 45-49.
[23] Bargmann, V. On a Hilbert space of analytic functions and an associated integral transform Part. 1 // Commun. Pure Appl. Math. – 1961. – Vol. 14. – P. 187-214.
[24] Namias, V. The fractional order Fourier transform and its application to quantum mechanics // J. Inst. Math. Appl. – 1980. – Vol. 25. – P. 241-265.
[25] McBride, A.C. On Namias' fractional Fourier transforms / A.C. McBride, F.H. Kerr // IMA J. Appl. Math. – 1987. – Vol. 39. – P. 159-265.
[26] Ozaktas, H.M. Fourier transform of fractional order and their optical interpretation / H.M. Ozaktas, D. Mendlovic // Opt. Commun. – 1993. – Vol. 110. – P. 163-169.
[27] Lohmann, A.W. Image rotation, Wigner rotation, and the fractional order Fourier transform // J. Opt. Soc. Am. A. – 1993. – Vol. 10. – P. 2181-2186.
[28] Ozaktas, H.M. The fractional Fourier transform / H.M. Ozaktas, Z. Zalevsky, M.A. Kutay // Chichester: Wiley, 2001. – 280 p.

Acknowledgments
This work was supported by the RFBR grant 17-07-00886 and by the Ural State Forest Engineering's Center of Excellence in «Quantum and Classical Information Technologies for Remote Sensing Systems.